ISRAEL JOURNAL OF MATHEMATICS 113 (1999), 285–304

AUBRY–MATHER THEORY AND THE INVERSE SPECTRAL PROBLEM FOR PLANAR CONVEX DOMAINS

BY

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ABSTRACT

The inverse spectral problem is concerned with the question to what extent the spectrum of a domain determines its geometry. We find that, associated to a convex domain Ω in $\mathbb{R}^2$, there is a convex *function* which is a length spectrum invariant under continuous deformations. It includes several geometric quantities, such as the lengths and Lazutkin parameters of caustics, as well as the asymptotic invariants discovered by Marvizi and Melrose. Via a Poisson relation, we also find invariants determined by the Laplace spectrum of Ω.

1. Introduction

Suppose $\Omega \subset \mathbb{R}^2$ is a strictly convex domain with smooth boundary $\partial\Omega$. The marked length spectrum $\mathcal{ML}(\Omega)$ is the function that associates to each rational number m/n the maximal length of closed geodesics in Ω having rotation number m/n. By geodesic we mean a billiard trajectory, in other words, the trajectory of a particle that moves freely inside Ω and gets reflected at $\partial\Omega$. The inverse spectral problem for $\mathcal{ML}(\Omega)$ is concerned with the question how much information about the geometry of Ω is encoded in the marked length spectrum.

One way to study this problem is to construct marked length spectrum invariants (MLS-invariants). These are geometric quantities which are the same for convex domains with the same marked length spectrum. Difficulties may arise

Received March 23, 1998

in a twofold way: to prove that a certain geometric quantity is a MLS-invariant, or to give a geometric meaning to some given MLS-invariant. There are several different definitions of MLS-invariants [MM, Am2, Po] whose mutual relation has not been clear.

The main objective of this paper is to establish a link between the inverse spectral problem for $\mathcal{ML}(\Omega)$ and what is known as Aubry–Mather theory in dynamical systems, where one investigates action-minimizing orbits of monotone twist mappings. It is a simple, but fruitful, observation that the marked length spectrum $\mathcal{ML}(\Omega)$ appears in the so-called mean minimal action functional α from Aubry–Mather theory. In fact, this theory provides a framework that comprises all MLS-invariants at once, and enables one to apply general results about α in the inverse spectral problem. Once this link is made, the fact that certain quantities are MLS-invariants becomes almost tautological. Moreover, several geometric parameters of Ω—such as its diameter, its boundary length, and parameters of convex caustics—are hidden in the mean minimal action.

Let us be more precise. The mean minimal action, associated to Ω, is a strictly convex function

$$\alpha\colon [0,1] \to \mathbb{R}$$

with convex conjugate

$$\alpha^*\colon [-1,1] \to \mathbb{R}.$$

A convex caustic $\mathfrak{c}$ is a closed convex C^1-curve in Ω with the property that every trajectory tangent to $\mathfrak{c}$ stays tangent after each reflection. The existence of a Cantor set of convex caustics near $\partial\Omega$ was proven by Lazutkin using KAM-theory [La]. Among our results are the following:

- α is smooth on a Cantor set containing the boundary points 0 and 1 (Theorem 4.1). The same holds true for $(\alpha^*)^{2/3}$ with boundary points -1 and 1.
- If $\mathfrak{c}$ is a convex caustic, then $-\alpha'$ is the length of $\mathfrak{c}$, and α^* its Lazutkin parameter (Theorem 4.3).
- The disk is characterized by the property that the corresponding mean minimal action is differentiable (Theorem 4.6).
- The countably many MLS-invariants ("integral invariants") by Marvizi and Melrose [mm] are algebraically equivalent to the Taylor coefficients of $(\alpha^*)^{2/3}$ at -1.

We have tried to keep our exposition as simple as possible. Therefore, Sections 2 and 3 contain concise reviews of the relevant definitions and facts concerning convex billiards and Aubry–Mather theory. The reader already familiar with these topics can go directly to Section 4; this is the main section of the paper and explains, among others, the results mentioned above. Finally, Section 5 recalls a well-known relation between the length spectrum and the Dirichlet spectrum and shows that, under a certain noncoincidence assumption on Ω, the Taylor coefficients of α at 0 are completely determined by the Dirichlet spectrum.

ACKNOWLEDGEMENT: This work was done when I was supported by a post-doctoral grant from the DFG–Graduiertenkolleg "Nichtlineare Differentialgleichungen" at the University of Freiburg, Germany. It originated in a question J. Moser asked several years ago. I would like to thank him, G. Popov and E. Zehnder for various discussions, as well as V. Bangert and L. Polterovich for reading and commenting on a preliminary version. These results were presented at ETH Zurich in December 1997, and I am grateful to H. Geiges and E. Zehnder for their invitation.

2. Convex billiards

Suppose we are given a smooth (i.e. C^∞) strictly convex domain Ω in $\mathbb{R}^2$. We will always assume that the length of its boundary $\partial\Omega$ is normalized to 1:

$$\partial\Omega \cong \mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}.$$

A closed geodesic in Ω is a broken geodesic in $\mathbb{R}^2$ that is reflected when it hits the boundary, according to the law "angle of reflection = angle of incidence".

The natural question arises whether there exist closed geodesics at all. In order to distinguish topologically different closed geodesics, we associate to each such geodesic its **rotation number**

$$\frac{m}{n} = \frac{\text{winding number}}{\text{number of reflections}} \in \left(0, \frac{1}{2}\right].$$

The winding number $m \geq 1$ is defined as follows. We fix the positive orientation of $\partial\Omega$ and pick any corner point of the closed geodesic on $\partial\Omega$. Then we follow the geodesic and measure how many times (with respect to the chosen orientation) we go around $\partial\Omega$, until we come back to the starting point. Note that we restrict ourselves to rotation numbers less than or equal to $\frac{1}{2}$, since a closed geodesic with rotation number ω can be seen as one with rotation number $1-\omega$, traversed in the backward direction.

THEOREM 2.1 (Birkhoff): *For every $m/n \in \left(0, \frac{1}{2}\right]$ in lowest terms, there exists a closed geodesic having rotation number m/n which, in addition, maximizes the length amongst all inscribed n-gons in Ω with winding number m.*

In order to understand the proof, we reformulate the problem. A broken geodesic in Ω is completely determined by its reflection points, together with the angles of reflection. The map

$$\phi: \mathbb{S}^1 \times (0,\pi) \to \mathbb{S}^1 \times (0,\pi),$$
$$(s_0,\psi_0) \mapsto (s_1,\psi_1),$$

that associates to a pair (s,ψ)=(arclength on $\partial\Omega$, angle with the positive tangent) the corresponding data at the next reflection, is called the **billiard map** associated to Ω. Let us denote by

$$h(s,s') = -|P(s) - P(s')|$$

the negative Euclidean distance between two points on $\partial\Omega$. Elementary geometry shows that

$$\partial_1 h(s_0,s_1) = \cos\psi_0 \quad \text{and} \quad \partial_2 h(s_0,s_1) = -\cos\psi_1$$

where ∂_i stands for the partial derivative with respect to the i-th variable. In other words, in new coordinates

$$(x,y) = (s, -\cos\psi)$$

we have

$$y_1\,dx_1 - y_0\,dx_0 = dh(x_0,x_1). \tag{1}$$

This means that ϕ is exact symplectic on $\mathbb{S}^1 \times (-1,1)$ with generating function h being the negative Euclidean distance.

Sketch of the proof of Theorem 2.1: Consider the length functional

$$H(s_0,\dots,s_n) = \sum_{i=0}^{n-1} |P(s_i) - P(s_{i+1})|$$

on the set of ordered $(n+1)$–tuples with $s_0 \le s_1 \le \cdots \le s_n = s_0 + m$. Such tuples parametrize all inscribed n-gons with vertices $P(s_0),\dots,P(s_{n-1}),P(s_n) = P(s_0) \in \partial\Omega$. They form a compact set, so the continuous function H has a maximum. We want to show that it does not lie on the boundary, which consists of degenerate n-gons with less than n vertices.

A critical point of H corresponds to a tuple such that

$$\partial_1 h(x_i,x_{i+1}) + \partial_2 h(x_{i-1},x_i) = 0. \tag{2}$$

Hence it gives an orbit (x_i, y_i) of the billiard map, where the y_i are determined by (1). But, since adding a vertex to a polygon in Ω will increase its perimeter, the above maximum cannot be represented by a degenerate n-gon. ∎

In fact, Theorem 2.1 is only the easy part of Birkhoff's theorem which guarantees the existence of at least two closed geodesics with rotation number m/n; the second one corresponds to a saddle point of the length functional. A proof can be found, for instance, in [KH, Sect. 9.3].

Theorem 2.1 allows one to define the **marked length spectrum** of Ω as the map

$$\mathcal{ML}(\Omega)\colon \mathbb{Q} \cap (0, \frac{1}{2}] \to \mathbb{R}$$

that associates to any m/n in lowest terms the maximal length of closed geodesics having n vertices and winding number m. In contrast, the **length spectrum** of Ω is the set

$$\mathcal{L}(\Omega) = \mathbb{N} \cdot \{\text{lengths of closed geodesics in } \Omega\} \cup \mathbb{N}.$$

It contains information about *all* closed geodesics, albeit in an "unformatted" form. The marked length spectrum, as a function, does give the labelling by the rotation number, but only for the closed geodesics of maximal length.

In general, it is not clear to what extent the marked length spectrum is determined by the length spectrum. For continuous deformations, however, one has the following result.

PROPOSITION 2.2: *Suppose Ω_s is a continuous family of strictly convex domains such that $\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)$ for all s. Then $\mathcal{ML}(\Omega_s) = \mathcal{ML}(\Omega_0)$ for all s.*

Sketch of proof: We have seen above that closed geodesics correspond to critical points of the length functional. Hence, by Sard's Theorem, $\mathcal{L}(\Omega_s) \subset \mathbb{R}$ has Lebesgue measure zero. The marked length spectra of Ω_s are functions, continuously varying with s, with values in a set of Lebesgue measure zero—hence they cannot depend on s. ∎

One way of investigating the (marked) length spectrum is to look for length spectrum invariants (**LS-invariants**), respectively, for marked length spectrum invariants (**MLS-invariants**). In particular, one would like to find such invariants that carry some geometric information about the domain Ω.

COROLLARY 2.3: *For continuous deformations of strictly convex domains, MLS-invariants are also LS-invariants.* ∎

In other words, if a certain quantity stays invariant under deformations with $\mathcal{ML}(\Omega_s) = \mathcal{ML}(\Omega_0)$, then it is also invariant under deformations with $\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)$.

A **convex caustic** is a closed convex C^1-curve in the interior of Ω with the property that a trajectory, that is tangent to it, stays tangent after each reflection. As an example, let us consider the disk: it is foliated by convex caustics which are just concentric circles.

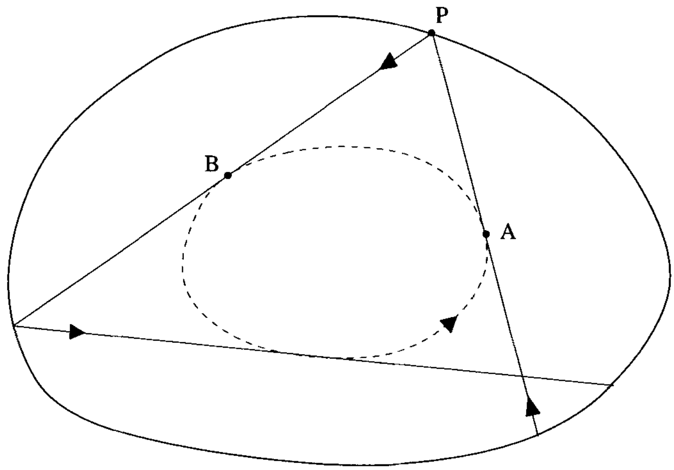


Figure 1. A convex caustic.

Given a convex caustic $\mathfrak{c}$, one can define the following parameters:

1. its rotation number $\omega \in (0, \frac{1}{2})$, defined as the rotation number of the circle homeomorphism on $\mathfrak{c}$ induced by the geodesic flow via the points of tangency;
2. its length $L(\mathfrak{c})$;
3. its **Lazutkin parameter** $Q(\mathfrak{c}) = |A - P| + |P - B| - |\overset{\frown}{AB}|$, with $A, B \in \mathfrak{c}$ and $P \in \partial\Omega$ as in Figure 1. Here $|\overset{\frown}{AB}|$ denotes the length of the caustic's part from A to B, where we have oriented the caustic according to the geodesics touching it.

If $\mathfrak{c}$ is a caustic, the Lazutkin parameter is well defined, i.e., it does not depend on the point $P \in \partial\Omega$ [La, Am1].

Are there always convex caustics in an arbitrary convex domain Ω? To study this question, it is convenient to formulate it in terms of the billiard map associated to Ω. Recall that ϕ: $(x_0, y_0) \mapsto (x_1, y_1)$ is an exact symplectic map on the open annulus $\mathbb{S}^1 \times (-1,1)$. Moreover, due to the strict convexity of Ω, it satisfies the so-called monotone twist condition

$$\frac{\partial x_1}{\partial y_0} > 0.$$

Thus, ϕ is what we call a monotone twist map; see Section 3. We may extend ϕ to $\mathbb{S}^1 \times [-1,1]$ by fixing the boundaries pointwise. More precisely, let us lift everything to the universal cover $\mathbb{R} \times [-1,1]$, and define

$$\phi(x,-1) = (x,-1) \quad \text{and} \quad \phi(x,1) = (x+1,1).$$

This reflects the fact that starting with an angle of almost π means having rotation number almost 1. The extended billiard map is a smooth area–preserving diffeomorphism on $\mathbb{R} \times (-1,1)$, but only a homeomorphism on $\mathbb{R} \times [-1,1]$.

To a convex caustic in Ω corresponds a **rotational invariant curve** for the billiard map, i.e. a simply closed, homotopically nontrivial curve Γ in $\mathbb{S}^1 \times (-1,1)$ with $\phi(\Gamma) = \Gamma$. The converse, however, is not entirely true. By a classical theorem of Birkhoff (cf. [Si2] and the references therein), rotational invariant curves are graphs and therefore do give rise to caustics; but these caustics need neither be convex nor smooth. Fortunately, caustics near the boundary are always smooth convex curves.

It turns out that, in appropriate coordinates, ϕ is a small perturbation of an integrable monotone twist map near $\{y=-1\}$. This observation enabled Lazutkin [La] to apply Moser's twist theorem [Mo] and prove the existence of a Cantor set of rotational invariant curves near $\{y=-1\}$. We will use a more refined version due to Kovachev and Popov [KP, Thm. 2].

THEOREM 2.4: *There are exact symplectic coordinates (θ, I) near $\{y=-1\} \leftrightarrow \{I=-1\}$, such that $\phi : (\theta_0, I_0) \mapsto (\theta_1, I_1)$ is generated by*

$$S(\theta_0, I_1) = \theta_0 I_1 + K(I_1)^{3/2} + R(\theta_0, I_1),$$

i.e.

$$\text{(3)} \qquad \begin{cases} I_0 = \partial_1 S = I_1 & +\partial_1 R, \\ \theta_1 = \partial_2 S = \theta_0 + \frac{3}{2} K(I_1)^{1/2} K'(I_1) & +\partial_2 R. \end{cases}$$

Here, $K \in C^\infty(\mathbb{R},\mathbb{R})$ with $K(-1) = 0, K'(-1) > 0$, and $R \in C^\infty(\mathbb{R}^2,\mathbb{R})$ is 1-periodic in the first variable. Moreover, there exists a Cantor set $C^* \subset [-1,-1+\epsilon^*)$ with $-1 \in C^*$, where $\epsilon^* > 0$ is some small number, such that $R \equiv 0$ on $\mathbb{R} \times C^*$.

We see that the perturbation term R vanishes on $\mathbb{R} \times C^*$ with all its derivatives. Each curve $\mathbb{R} \times \{I\}, I \in C^*$, gives rise to a rotational invariant curve for the billiard map, on which it is conjugated to the rigid rotation

$$(\theta, I) \mapsto (\theta + \omega, I) \tag{4}$$

with $\omega = \frac{3}{2} K(I)^{1/2} K'(I)$. Since these invariant curves lie near the boundary, they correspond to a Cantor set of convex caustics near and accumulating at $\partial\Omega$.

3. Aubry–Mather theory

This section deals with general monotone twist mappings on an annulus or a (half) cylinder. We will not distinguish between mappings on $\mathbb{S}^1 \times (a, b)$ and lifts to $\mathbb{R} \times (a, b)$. By definition, a **monotone twist map** is a smooth diffeomorphism

$$\phi : \mathbb{S}^1 \times (a, b) \to \mathbb{S}^1 \times (a, b),$$
$$(x_0, y_0) \mapsto (x_1, y_1),$$

where $-\infty \leq a < b \leq \infty$, satisfying the following conditions:

1. ϕ preserves orientation and the ends of $\mathbb{S}^1 \times (a, b)$, in the sense that $y_1(x_0, y_0) \to a$ as $y_0 \to a$. If a or b is finite, we require that ϕ can be extended to a homeomorphism on the closure of $\mathbb{S}^1 \times (a, b)$, again denoted by ϕ, such that $\phi(x, a) = (x + \omega_-, a)$ and $\phi(x, b) = (x + \omega_+, b)$, respectively.
2. ϕ is exact symplectic, i.e., there is a generating function h such that
$$y_1 \, dx_1 - y_0 \, dx_0 = dh(x_0, x_1).$$
3. ϕ satisfies the monotone twist condition
$$\frac{\partial x_1}{\partial y_0} > 0.$$

According to the four possible cases, we call $(-\infty, \infty), (-\infty, \omega_+], [\omega_-, \infty)$, or $[\omega_-, \omega_+]$, the **twist interval** of ϕ; abusing notation, we denote it by $[\omega_-, \omega_+]$.

Monotone twist mappings are not as artificial as they seem. They are hidden in a variety if situations; see [Ba1, KH, MF] for examples and further references.

Example: We learned in the last section that the billiard map ϕ, associated to a strictly convex domain Ω in $\mathbb{R}^2$, is a monotone twist map on $\mathbb{S}^1 \times (-1, 1)$ with twist interval

$$[\omega_-, \omega_+] = [0, 1].$$

Theorem 2.4 revealed an abundance of rotational invariant curves for ϕ near the two boundary components $\{y = \pm 1\}$. On each of these invariant curves near $\{y = -1\}$, ϕ is conjugated to a rotation by some angle ω in the Cantor set

$$\mathcal{C} = \left\{ \frac{3}{2} K(I)^{1/2} K'(I) | \; I \in \mathcal{C}^* \right\} \subset [0, \epsilon) \tag{5}$$

containing 0. If we had rotational invariant curves for all rotation numbers $\omega \in [0,1]$, the billiard table Ω would necessarily be a disk, due to a recent theorem by Bialy [Bi]. Thus we cannot expect "too many" of them. ∎

There are, however, always ϕ-invariant *sets* of any prescribed rotation number, and these sets resemble rotational invariant curves insofar as they lie on graphs over the x-axis. This is, very roughly, the content of a theory, developed independently by Aubry and Mather. The interested reader may read the article by Bangert [Ba1], or the corresponding chapters in [KH] or [MF]. In this section, we will just review certain ideas very briefly and collect the facts we will need later.

The starting point for the theory is Theorem 2.1, which shows the existence of closed geodesics that maximize the perimeter. Recall that maximizing length means minimizing the **action**, defined as the sum of $h(x_i, x_{i+1})$ along the orbit. This leads to the following variational principle connected with a generating function h.

An infinite sequence $(s_i)_{i\geq 0}$ of real numbers is called **minimal**, if every finite segment $(s_k, \dots, s_{k+l})$ minimizes the action amongst all $(l-k+1)$-tuples with the same end points; in other words, if

$$\sum_{i=k}^{k+l-1} h(s_i, s_{i+1}) \leq \sum_{i=k}^{k+l-1} h(\xi_i, \xi_{i+1})$$

for all $(\xi_k, \dots, \xi_{k+l})$ with $\xi_k = s_k$ and $\xi_{k+l} = s_{k+l}$. An orbit $(x_i, y_i)_{i\geq 0}$ of a monotone twist map ϕ is called minimal, if the sequence $(x_i)_{i\geq 0}$ is minimal, with h being a (and hence any) generating function of ϕ.

Meanwhile, we have two notions of minimality—the one used in Theorem 2.1 for minimizing the action among all *periodic* orbits, and the one above. It is a special feature in dimension 2, that these two notions coincide. In particular, minimal (m,n)-periodic orbits are also minimal (km, kn)-periodic orbits for $k \geq 1$. Thus, a refined version of Theorem 2.1, with the length argument in the end adapted to the setting of general monotone twist maps, yields the existence of minimal periodic orbits of every rational rotation number m/n. Taking limits of such minimal (m_j, n_j)-periodic orbits of "Birkhoff type", where $m_j/n_j \to \omega$ converges to an irrational, one can prove the existence of minimal orbits (x_i, y_i) for any given rotation number $\omega = \lim_{i\to\infty}(x_i/i)$.

THEOREM 3.1 (Aubry, Mather): *Suppose ϕ is a monotone twist map, and ω is in its twist interval. Then the set of minimal orbits having rotation number ω is non-empty and, moreover, lies on a Lipschitz graph.*

Besides, perhaps the most direct proof for the existence of orbits of all rotation numbers was given by Golé [Go]. He considers the gradient flow of the action "functional" $\sum_{i=0}^{\infty} h(s_i, s_{i+1})$, restricted to certain subspaces of sequences with rotation number ω. In view of (2), rest points of this flow will give true orbits. Their minimality, however, has to be shown separately.

Since we have minimal orbits of all rotation numbers, we may define the **mean minimal action**

$$\alpha\colon [\omega_-, \omega_+] \to \mathbb{R}$$

by associating to ω the mean action of a minimal orbit (x_i, y_i) of rotation number ω:

$$\alpha(\omega) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} h(x_i, x_{i+1}).$$

This is well defined, i.e., the limit exists and does not depend on the particular minimal orbit (which is obvious for periodic orbits).

In the next section, we will need the following properties of the mean minimal action.

PROPOSITION 3.2: *α is a strictly convex function.*

See [Ma1, §4] or [Ma3, Sect. 6] for a proof. In particular, α is continuous. Concerning its differentiability, we have the following result.

THEOREM 3.3: *Suppose ϕ is a monotone twist map with $\omega \in [\omega_-, \omega_+]$. Then the following statements hold:*

1. *If ω is irrational then α is differentiable at ω.*
2. *If there is a rotational invariant curve Γ_ω of rotation number ω, then α is differentiable at ω with $\alpha'(\omega) = \int_{\Gamma_\omega} y\, dx$.*

Proof: The differentiability properties of α in both parts were proven by Mather [Ma2]. If Γ_ω is a rotational invariant curve then each orbit on it has the same rotation number ω, and is, in fact, a minimal orbit [MF, Thm. 17.4]. The latter can also be shown in the more general context of action minimizing measures in arbitrary dimensions, if one uses Moser's result that every monotone twist map can be interpolated by a convex Hamiltonian. Then [Si1, Thm. 2.1] gives $\int_{\Gamma_\omega} y\, dx$ as the derivative of α at ω. ∎

4. The mean minimal action and MLS-invariants

In this section, we will put general results from Aubry–Mather theory into the context of billiard maps. Our emphasis lies on the study of one central object: the mean minimal action α. On the one hand, α can be interpreted as a dynamical version of the marked length spectrum and, as such, comprises all possible MLS-invariants. Therefore, all MLS-invariants do have a common root. On the other hand, we will also see that a lot of geometric information about Ω is hidden in the function α.

Recall from Section 2 that the billiard map associated to a strictly convex domain Ω is generated by h, the negative distance between points on $\partial\Omega$. In the language of Aubry–Mather theory, minimal orbits correspond to geodesics of maximal length. As before, we assume that all domains under consideration have unit boundary length.

THEOREM 4.1: *The mean minimal action for the billiard map associated to a strictly convex planar domain Ω is a complete MLS-invariant, i.e., $\mathcal{ML}(\Omega_1) = \mathcal{ML}(\Omega_2)$ if and only if $\alpha_1 = \alpha_2$. It is a strictly convex function on $[0,1]$, which is smooth on a Cantor set containing the points 0 and 1.*

More precisely, the proof shows that α, restricted to that Cantor set near 0, can be extended to a smooth, odd function on $\mathbb{R}$.

Proof: The first assertion follows from the continuity of α and the fact that $\alpha(m/n) = -\frac{1}{n}\mathcal{ML}(\Omega)(m/n)$ for every $m/n \in (0, \frac{1}{2}]$ in lowest terms. The strict convexity is contained in Proposition 3.2. For the last part, consider the convex conjugate function

$$\alpha^*\colon [-1,1] \to \mathbb{R}$$

defined by

$$\alpha^*(I) = \max_{\omega}\,[\omega I - \alpha(\omega)].$$

By standard results in convex analysis [MW, Ch. 2], α^* is a C^1-function; moreover, if α is differentiable at ω, we have

$$(\alpha^*)'(\alpha'(\omega)) = \omega.$$

Hence, in view of (4), $\theta_0 I_1 + \alpha^*(I_1)$ generates $\phi : (\theta_0, I_0) \mapsto (\theta_1, I_1)$ on the Cantor set $\mathbb{R} \times C^*$ of invariant curves, accumulating at $\mathbb{R} \times \{-1\}$. Besides, this explains the domain of definition for α^* given above.

We also know from Theorem 2.4 that ϕ is generated by $S = \theta_0 I_1 + K(I_1)^{3/2} + R(\theta_0, I_1)$, where K is smooth and R vanishes with all its derivatives on $\mathbb{R} \times C^*$.

Since $\alpha^*(-1) = 0 = K(-1)$, we must have that

$$\alpha^*(I) = K(I)^{3/2} \tag{6}$$

for all $I \in \mathcal{C}^*$. The smooth function K can be written as

$$K(I) = K'(-1) \cdot (I+1) + O((I+1)^2) \tag{7}$$

with $K'(-1) > 0$. Combining (6) and (7), we obtain that

$$\alpha(\omega) = a_1\omega + a_3\omega^3 + O(\omega^5)$$

is smooth on the Cantor set $\mathcal{C}$ containing 0. Hence it can be extended to an odd smooth function on $\mathbb{R}$.

By symmetry, analogous statements hold near $I = 1$ respectively $\omega = 1$. ∎

Surprisingly, it seems that the concavity of the function $\frac{m}{n} \mapsto \frac{1}{n}\mathcal{ML}(\Omega)(\frac{m}{n})$, as well as the smoothness property of $\mathcal{ML}(\Omega)$, do not appear in the literature.

COROLLARY 4.2: *For continuous deformations of strictly convex domains, α is an LS-invariant function on* $[0,1]$, *which is smooth on a Cantor set containing the points* 0 *and* 1.

Proof: Immediate from Corollary 2.3. ∎

By way of illustration, let us calculate the simplest example.

Example: Take Ω to be the disk of perimeter 1 in $\mathbb{R}^2$. The billiard map is given by $(s_1, \psi_1) = (s_0 + \psi_0/\pi, \psi_0)$ with generating function

$$h(s,s') = -\frac{1}{\pi}\sin\pi(s-s').$$

The whole phase space is foliated by rotational invariant curves $\{y = \text{const.}\}$, and the mean minimal action is just

$$\alpha(\omega) = -\frac{1}{\pi}\sin\pi\omega.$$

In view of the identity $\sin\arccos(-x) = \sqrt{1-x^2}$, its convex conjugate is

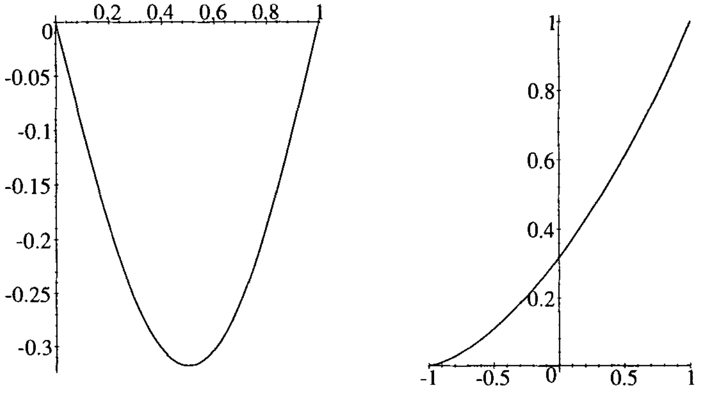
$$\alpha^*(I) = \frac{1}{\pi}\left(\arccos(-I)\cdot I + \sqrt{1-I^2}\right) \in [0,1]$$

for $I \in [-1,1]$; see Figure 2. The asymptotics for $\omega \to 0$ and $I \to -1$, respectively, are as follows:

$$\alpha(\omega) = -\omega + \frac{\pi^2}{6}\omega^3 - \frac{\pi^4}{120}\omega^5 + O(\omega^7),$$

$$\alpha^*(I) = \frac{\sqrt{2}}{\pi}\left(\frac{2}{3}(I+1)^{3/2} + \frac{1}{30}(I+1)^{5/2} + \frac{3}{560}(I+1)^{7/2}\right) + O((I+1)^{9/2}).$$

Note that α is an odd smooth function on $\mathbb{R}$. α^* has a singularity of order $\frac{3}{2}$ at $I = -1$ so that the function $K = (\alpha^*)^{2/3}$ from (6) is smooth. ∎



(a) The mean minimal action α (b) Its convex conjugate $\alpha*$

Figure 2. The circular billiard.

In the following, we consider relations between the geometric parameters of convex caustics and analytical quantities of the mean minimal action. The next theorem gives a concrete geometric interpretation for the derivative and the convex conjugate.

THEOREM 4.3: *Suppose $\mathfrak{c}_\omega$ is a convex caustic of rotation number ω. Then the length of $\mathfrak{c}_\omega$ and its Lazutkin parameter are given by $L(\mathfrak{c}_\omega) = -\alpha'(\omega)$ and $Q(\mathfrak{c}_\omega) = \alpha^*(\alpha'(\omega))$, respectively.*

Proof: Call $T(s)$ and $N(s)$ the unit tangent vector and unit inward normal, respectively, at a point $P(s) \in \partial\Omega$, and set $U(s) = \cos\psi(s)T(s) + \sin\psi(s)N(s)$; here, $\psi(s) \in (0, \pi/2)$ is the unique angle such that the ray from $P(s)$ having direction $U(s)$ touches $\mathfrak{c}_\omega$. Then there is a smooth function $\tau(s)$ such that

$$A(s) = P(s) + \tau(s)U(s) \in \mathfrak{c}_\omega$$

and

$$\dot{A}(s) = T(s) + \dot{\tau}(s)U(s) + \tau(s)\dot{U}(s) \parallel U(s).$$

Since $\dot{A}(s) \neq 0$, we can write

$$|\dot{A}(s)| = \langle \dot{A}(s), U(s) \rangle = \cos\psi(s) + \dot{\tau}(s),$$

so that

$$L(\mathfrak{c}_\omega) = \int_{\mathbb{S}^1} |\dot{A}|\, ds = \int_{\mathbb{S}^1} \cos\psi\, ds = -\int_{\Gamma_\omega} y\, dx = -\alpha'(\omega)$$

by Theorem 3.3.

By definition of the Lazutkin parameter (see Figure 1), we have for $N \geq 1$

$$(N+1)Q(\mathfrak{c}_\omega) = |A_1 - P_1| + \sum_{i=1}^{N} |P_i - P_{i+1}| + |P_{N+1} - B_{N+1}| - \sum_{i=1}^{N+1} |\, \widehat{A_i B_i} \,|.$$

Hence

$$\begin{aligned} Q(\mathfrak{c}_\omega) &= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} |P_i - P_{i+1}| - \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} |\, \widehat{A_i B_i} \,| \\ &= -\alpha(\omega) - \omega L(\mathfrak{c}_\omega) \\ &= \omega\alpha'(\omega) - \alpha(\omega) \\ &= \alpha^*(\alpha'(\omega)). \qquad \blacksquare \end{aligned}$$

As a corollary, we obtain the main result in [Po]. Note, however, that the two components of the vector valued function in Corollary 4.4 are not independent.

COROLLARY 4.4: *The function $\omega \mapsto (L(\mathfrak{c}_\omega), Q(\mathfrak{c}_\omega))$, defined on a Cantor set in $[0,1]$ containing 0 and 1, is an MLS–invariant, respectively, an LS-invariant under continuous deformations.* ∎

In view of Theorem 4.3, one might call $-\alpha'$ and α^* the generalized "length" and "Lazutkin parameter", even if there is no convex caustic of the corresponding rotation number.

Theorem 4.3 also implies a functional dependence of $L(\mathfrak{c})$ and $Q(\mathfrak{c})$.

COROLLARY 4.5: *There is a formal power series expansion*

$$L = 1 + \sum_{k\geq 1} b_k Q^{2k/3}$$

as $Q \to 0$, whose coefficients are MLS-invariants.

Proof: $L(\mathfrak{c}_\omega) = -\alpha'(\omega)$ and $Q(\mathfrak{c}_\omega)^{2/3} = \alpha^*(\alpha'(\omega))^{2/3} = K(\alpha'(\omega))$ are smooth functions on $\mathcal{C}$ with $K'(-1) > 0$. The claim follows from the implicit function theorem. ∎

Corollary 4.5 was first stated explicitly in [Am2, (3.1)]. It follows, however, already from formulae obtained by Lazutkin, namely (1.11) and (1.12) in [La], where he expresses the length and his parameter in terms of the rotation number.

Given a convex caustic $\mathfrak{c}$, one can reconstruct $\partial\Omega$ by wrapping a string of length $L(\mathfrak{c}) + Q(\mathfrak{c})$ around $\mathfrak{c}$, pulling it tight, and going along $\mathfrak{c}$. These "string length parameters" of convex caustics are, of course, also MLS–invariants of the domain Ω.

An old conjecture, usually attributed to Birkhoff, states that the only integrable billiards are ellipses.* We have the following weaker rigidity result, which shows once more how analytical properties of α can translate into geometric properties of Ω.

THEOREM 4.6: *Suppose $\Omega \subset \mathbb{R}^2$ is a strictly convex domain, whose mean minimal action α is differentiable everywhere. Then Ω is a disk.*

Proof: According to [Ma2], the differentiability of α at a rational is equivalent to the existence of a rotational invariant curve consisting of minimal periodic orbits. Taking limits of these curves, we obtain rotational invariant curves for all rotation numbers.

We claim that they foliate the phase space. Indeed, if there was a gap, its boundary curves would necessarily have the same rotation number (otherwise, there would be rotation numbers without invariant curves). But this is impossible, due to the graph property of the set of minimal orbits in Theorem 3.1.

Now the theorem follows from Bialy's result [Bi] that the only billiard, whose phase space is foliated by rotational invariant curves, is a circular one. ∎

COROLLARY 4.7: *Suppose Ω_s is a continuous deformation of strictly convex domains, such that $\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)$ and Ω_0 is a disk. Then Ω_s is a disk for all s.*

Proof: In view of Corollary 4.2, α_s does not depend on s; in particular, α_s is differentiable everywhere because α_0 is. ∎

This result has an analogue in differential geometry [Ba2, Thm. 6.1]: a Riemannian 2-torus, having the same marked length spectrum as a flat torus, is flat.

Let us now turn to the integral invariants by Marvizi and Melrose [MM] mentioned in the introduction, and see how they fit into this framework. By definition, an interpolating Hamiltonian is a smooth function ζ on $\mathbb{S}^1 \times [-1, 1]$, whose time–$\zeta^{1/2}$–map is the billiard map, up to a diffeomorphism that fixes the

* Recently, Amiran [Am3] has proposed a proof of this, but I have not checked the details.

boundary to infinite order. The integral invariants are then defined as the Taylor coefficients of

$$J(r) = \frac{1}{\zeta'(\zeta^{-1}(r))}$$

at $r = 0$ for any interpolating Hamiltonian ζ in action-angle-variables. From (3) and (6), we obtain that $(\frac{3}{2})^{2/3}K(I) = (\frac{3}{2}\,\alpha^*(I))^{2/3}$ is such an interpolating Hamiltonian, at least on a Cantor set containing the boundary. Hence the integral invariants are algebraically equivalent to the Taylor coefficients of $(\alpha^*)^{2/3}$. By Theorem 4.1, they are MLS-invariants.

In order to prove the asymptotic formula from [MM] for the lengths of closed geodesics, we consider a closed geodesic g_{mn} of rotation number m/n. Its length is given by

$$l(g_{mn}) = -\sum_{g_{mn}} h. \tag{8}$$

We want to rewrite this in (θ, I)-coordinates and relate it to the generating function $S(\theta_0, I_1)$ from Theorem 2.4. We have the following transformations:

$$\Phi\colon (\theta, I) \mapsto (x,y) \quad \text{with } \Phi^*(y\,dx) - I\,d\theta = dH,$$
$$\Psi = \Phi^{-1} \circ \phi \circ \Phi,$$
$$\theta_1\,dI_1 + I_0\,d\theta_0 = dS \quad \text{with } S = \theta_0 I_1 + K(I_1)^{3/2} + R(\theta_0, I_1).$$

A straightforward calculation shows that the generating function transforms according to the formula

$$\Psi^*(I\,d\theta) - I\,d\theta = d(h\circ\Phi + H - H\circ\Psi). \tag{9}$$

On the other hand, we can write the left hand side as

$$\begin{aligned} I_1\,d\theta_1 - I_0\,d\theta_0 &= (-\theta_1\,dI_1 - I_0\,d\theta_0) + (\theta_1\,dI_1 + I_1\,d\theta_1) \\ &= -dS + \Big(\frac{\partial S}{\partial I_1}\,dI_1 + I_1\,d(\frac{\partial S}{\partial I_1})\Big) \\ &= d\Big(I_1\,\frac{\partial S}{\partial I_1} - S\Big) \\ &= dS^*. \end{aligned}$$

From this we conclude that

$$-h\circ\Phi = H - H\circ\Psi - S^* + \text{const}.$$

Summed over a closed orbit, the term $H - H \circ \Psi$ adds to zero, so (8) yields

$$\frac{1}{n}\, l(g_{mn}) = \frac{1}{n} \sum_{g_{mn}} -S^* + \frac{\text{const.}}{n}.$$

Since

$$S^* = (K(I)^{3/2})^* + I\, \frac{\partial R}{\partial I} - R$$

has the same Taylor series as the boundary point as $(K(I)^{3/2})^*$, which coincides on $\mathbb{S}^1 \times C$ with $(\alpha^*)^* = \alpha$, we see that $\frac{1}{n}\, l(g_{mn})$ has the same Taylor series for $n \to \infty$ as $-\alpha$. Loosely speaking, all closed geodesics become minimal as they approach $\partial\Omega$. More precisely,

$$l(g_{mn}) = m + \sum_{k \geq 1} c_{mk} n^{-2k}$$

where

$$c_{mk} = -\frac{m^{2k+1}}{(2k+1)!}\, \alpha^{(2k+1)}(0). \tag{10}$$

The formula shows that the asymptotics of the length spectrum are completely determined by the Taylor coefficients of the mean minimal action, and vice versa.

Another application of Theorem 4.1 concerns regions in Ω which are free of convex caustics. Gutkin and Katok gave estimates for their area in terms of the geometry of Ω. In particular, they proved [GK, Prop. 1.3] that a convex caustic $\mathfrak{c}_\omega$ cannot lie too far from the boundary:

$$\max_{P \in \mathfrak{c}_\omega} d(P, \partial\Omega) < \sqrt{\operatorname{diam}\Omega \cdot Q(\mathfrak{c}_\omega)}.$$

Surprisingly, the bound on the right hand side has an MLS-invariant interpretation:

$$\max_{P \in \mathfrak{c}_\omega} d(P, \partial\Omega) < \sqrt{-\alpha\big(\tfrac{1}{2}\big) \cdot \alpha^*(\alpha'(\omega))}. \tag{11}$$

We can formulate the following result.

PROPOSITION 4.8: *Suppose Ω_s is a continuous deformation of strictly convex domains such that $\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)$. Then, for a fixed rotation number ω, every convex caustic $\mathfrak{c}_\omega(s)$ in Ω_s is contained in a strip around $\partial\Omega_s$ of fixed width $\sqrt{-\alpha(1/2) \cdot \alpha^*(\alpha'(\omega))}$.* ∎

Finally, we point out that the mean minimal action is actually invariant under arbitrary *symplectic* coordinate changes in the phase space. For, by (9), the actions of periodic orbits are symplectically invariant, and they determine α.

5. Spectral invariants

As already mentioned in the introduction, there is a relation between the length spectrum $\mathcal{L}(\Omega)$ of a strictly convex domain and the spectrum of the Laplacian with Dirichlet boundary conditions:

$$\begin{cases} \triangle u = \lambda^2 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Namely, due to a Poisson relation for compact Riemannian manifolds with geometrically convex boundary [GM], the expression

$$\sigma(t) = \sum_{\lambda^2 \in \mathrm{spec}\triangle} \cos \lambda t$$

is well defined as a distribution, which is smooth away from $\mathcal{L}(\Omega)$ [AM]. More precisely, if $T > 0$ is in the singular support of σ then T lies in the length spectrum of Ω.

Conversely, whether some given $T \in \mathcal{L}(\Omega)$ belongs to the singular support of σ depends on possible cancellations of singularities stemming from different closed geodesics of the same length. It is known [GM] that $T \in$ sing.supp. σ if there is exactly one closed geodesic of length T, whose Poinaré map does not have an eigenvalue 1. Marvizi and Melrose [MM] showed that a much weaker noncoincidence condition on Ω suffices to conclude that almost all *maximal* lengths of geodesics having rotation number $1/n$ lie in sing.supp. σ. Popov [Po] generalized this to geodesics of rotation number m/n with $m > 1$, provided (m, n) is "near" the Cantor set described in (5). In particular, that noncoincidence condition is satisfied by all curvature functions in a C^1-neighbourhood of the constants.

Thus, the values $\alpha(1/n)$, respectively $\alpha(m/n)$, with sufficiently large n are spectral invariants of the domain. This is also true for the coefficients c_{1k} in (10), and hence for the Taylor coefficients of $-\alpha$ at 0. Therefore we can state the following result, which is an analogue of [MM, Thm. 7.4].

THEOREM 5.1: *Suppose $\Omega \subset \mathbb{R}^2$ is a strictly convex domain with unit boundary length, such that 1 is not a limit point of lengths of closed geodesics having fixed rotation number m/n with $m > 1$. Then the Taylor series of the mean minimal action at 0 is completely determined by the Dirichlet spectrum (12).*

Applying Popov's more general result, one can show that also the values of α on the Cantor set $\mathcal{C}$ (and hence all the caustic parameters $L(\mathfrak{c})$ and $Q(\mathfrak{c})$) are spectral invariants under the noncoincidence condition in [Po, (6.1)].

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